Computing the added mass of dispersed particles

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ABSTRACT

This paper investigates the possibility of using computational fluid dynamics (CFD) to evaluate the added mass coefficient \( C \) of dispersed particles, both individually and in clusters. Using the direct numerical simulation (DNS) approach, the volume of fluid (VOF) model was employed, as implemented in Fluent software, to move the particles. \( C \) was calculated from the initial acceleration of a buoyant particle released from rest. The acceleration data were evaluated from very short initial time intervals (typically \( \sim 10^{-5} \) s), in which buoyancy and added mass forces are predominant. The numerical parameters of the code were optimized in cases of known solutions for \( C \) so that other situations could be analyzed. Several configurations were calculated for their number of particles, shape and spatial arrangement.

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1. Introduction

The concept of added mass is denoted by AM, one of the classic achievements of theoretical fluid dynamics, contrasts the acceleration caused by a given force \( F \), of a particle of mass \( m_p \) in a vacuum and in a fluid. In a vacuum Newton’s law states that \( m_p \dot{a} = F \), while in a fluid it holds that \( m_p \dot{a} + m_a \dot{a} = F \) where \( \dot{a} = a - a_q \) is the added mass coefficient. The problem here is to determine \( C \).

Thus far the simplest case has been considered, in which the particle is of a simple shape (sphere) and translates without rotation in an unbounded inviscid fluid. In this case, the quantity \( C \) is a scalar and takes the value of 0.5. When the particle has a lower degree of symmetry and is allowed to rotate, we have a second-order tensor \( C \) with 6 components (6 \( \times \) 6 matrix). Matrix \( C \) acts on the vector \( a \) of (linear and angular) acceleration to produce vector \( \dot{F} \) of (force + torque), \( \dot{F} = - (\rho V^2 a + \hat{C} a) \) Consequently, \( a \) and \( F \) are generally misaligned. The components \( C_{ij} \) termed inertia coefficients (analogous to the concept of inertia tensor used in rigid body motion) correspond to the inertia effects acting in different directions of particle translation (forces) and rotation (torques). They are determined by the geometry of the problem: particle shape, particle configuration and presence of boundaries. Since they are not dependent on the physical (material) nature of the particles, these coefficients apply, without discrimination, to bubbles, drops and solids. Tensor \( \hat{C} \) may be simplified by the presence of symmetry and by the choice of a suitable coordinate system.

The theoretical approach to AM has been described in a number of advanced texts on fluid mechanics (e.g., Batchelor, 1967; Birkhoff, 1960; Lamb, 1932; Milne-Thomson, 1968; Robertson, 1963; Yih, 1988), as well as in those focusing on fluid inertia forces (Newman, 1977). In addition, analytical and experimental results for the AM coefficients of various bodies (e.g. rectangular shapes, rotating ellipsoids, etc.) can be found in Brennen (1982).

Much has also been written about the practical applications of AM. As AM is an important issue when particles move in fluids with unsteady motion, one area of naval research focuses on the strong forces of inertia experienced by floating and underwater objects.
Multiphase fluid dynamics is another area of research in which dispersed particles are also exposed to these forces (Clift et al., 1978; Michaelides, 2006; Wallis, 1990). Here, the simplest case is a single dispersed particle translating in an unbounded domain. Two factors complicate this situation: the presence of domain boundaries and the presence of other particles. We need C for the following typical situations: particles approaching a wall, two and more particles in certain configurations (doublets, triplets, arrays, etc.), expanding/collapsing particles, oscillating particles, rotating particles, etc. Some of these simpler situations have already been solved, most of them having been approached analytically from the perspective of potential flow theory. In addition, experiments have been performed on the forces acting on oscillating bodies, where the period depends on C. Although C has rarely been obtained using computational fluid dynamics (CFD), we believe that knowledge about C is scaleable from the microscale (simple situations: single particle, pairwise interactions, etc.) to the macroscale, where the effective inertia forces acting within a multiphase mixture are estimated using various averaging methods. This forms the basis for modelling inertia coupling between various phases.

In recent decades, in the chemical engineering community the concept of AM has become the subject of more research. Primarily, this is due to the surge of interest in CFD, where it is now apparent that AM forces play an important role in the dynamics of dispersed systems (flow patterns, hydrodynamic stability, etc.). AM forces may be relevant in cases where the density of the dispersed (particulate) phase \( \rho_p \) is comparable to, or even lower than, that of the continuous (carrying) phase \( \rho \). This applies to liquid–liquid (drops in liquids) and solid–liquid (solids in liquids) systems and, in particular, to gas–liquid systems (bubbles in liquids) where, because the interphase density ratio \( \rho/\rho_p \) is \( \sim 10^3 \), almost all the inertia is concentrated in the carrying liquid phase.

AM also plays an important role in the hydrodynamics of bubble columns, which is our primary research interest. As this is such a complex problem, few experimental and analytical studies have been conducted, and most of the data in this area have been obtained using multiphase CFD simulations (see e.g., Jakobsen, 2005; Joshi, 2001; Ranade, 2002; Prosperetti and Tryggvason, 2007). Delnoij et al. (1997) carried out a Euler–Lagrange simulation of a partially aerated bubble column and concluded that AM plays an important role in the vicinity of the gas distributor. While performing linear stability analysis of a 1D Euler–Euler (E/E) model of a bubble column, Leon-Becerril and Line (2001) found that AM has a significant effect on homogeneous–heterogeneous flow regime transition. Subsequently, Leon-Becerril et al. (2002), carrying out 3D E/E simulations of a partially aerated rectangular bubble column, found it necessary to include AM in the model in order to reproduce bubble plume oscillations (data published by Becker et al., 1994). The same authors simulated mixing in a fully aerated rectangular column, again reporting the importance of AM. Monahan et al. (2005) observed that AM force is important in stabilizing homogeneous regimes, and cannot be neglected in large columns. In the model for homogeneous–heterogeneous regime transition in bubble columns developed by Bhole and Joshi (2005) the AM coefficient was a relevant control parameter. AM can also affect the values of the terms describing the production of turbulent kinetic energy due to the presence of gas–liquid interfaces (e.g., Mudde and Simonin, 1999; Joshi, 2001). Chahed et al. (2003) introduced turbulence correlations related to AM into their expression for interphase force, and found them to be of great importance in the computation of the void fraction in non-homogeneous bubbly flows. Various experimental studies have also been conducted (e.g., Cai and Wallis, 1993; Kendouh et al., 2007; Odar and Hamilton, 1964; Takahashi and Endoh, 1992).

Other results have also shown the ambiguity of the problem of AM. Mudde and Simonin (1999) concluded that it is necessary to use AM in E/E simulations of bubble plume with a given CFD program. However, different codes showed less need for AM (Oey et al., 2003), oscillatory plume being reproducible without factoring in AM. Deen et al. (2001) and Zhang et al. (2006) reported that AM forces had negligible effect on their E–E simulation results for a centrally aerated rectangular bubble column (except at the very top and bottom of the column). Likewise, Tabib (2007) reported that AM had no significant influence on the results they obtained using a 3D transient simulation of a cylindrical bubble column.

It is, therefore, clear that AM is an important parameter in the description of multiphase systems, and can even play the dominant role in certain specific flow situations taking place in technologically important processes. AM particularly applies in those situations where the density ratio \( \rho/\rho_p \) is large, the motion unsteady, the particle shape and configuration liable to change and the presence of boundaries influential. Thus, in this paper we use CFD to explore the possibility of determining AM in a variety of important situations. In each case, the value of C is obtained from the initial acceleration of the particle(s); this being a relatively simple and efficient way of evaluating C in situations that are well beyond the power of analytical approaches and inaccessible to experimentation. Our approach was inspired by the work of Niemann and Laurien (2001) and Laurien and Niemann (2004), who, to the best of our knowledge, were the first to use this method for the calculation of C.

### 2. Evaluation of C

Consider a buoyant particle, \( (\rho_p < \rho) \) released from rest in a stagnant fluid. During a very short initial time interval two forces act upon it: the buoyancy force, \( F_b = g(\rho - \rho_p)V \), is a constant and accelerates it upwards; the AM force, \( F_a = m_p a = C g V a \), is proportional to the initial acceleration \( a \) and opposes the motion. This remains approximately true until other forces develop, namely, those depending on body speed (e.g., drag) and until the possible body deformation is negligible. Upon substitution, Newton’s law of force

\[
m_p a = F_b - F_a
\]

becomes

\[
(\rho_p + C \rho) V a = g(\rho - \rho_p) V.
\]

The formula for the coefficient is thus:

\[
C = g/a - \rho_p / \rho
\]

where \( g \) is the reduced gravity, \( g(\rho - \rho_p)/\rho \), and \( a \) the initial bubble acceleration determined by CFD.

Our buoyant particles, or bodies, were gas bubbles in liquids. CFD simulations of free bubble rise were performed for a very short time interval, \( \sim 10^{-3} \) s. Due to the parabolic nature of the hydrodynamic equation (Eq. (3.2)), the disturbance produced by the accelerating body propagates through the whole flow domain so quickly that even such a short time interval is sufficient for the development of the flow field. The volume of fluid (VOF) model, as implemented in Fluent software, was used to perform the simulations. The bubble speed \( v \) was obtained using a user defined function to take ten measurements within each time interval. The value of \( a \) was obtained by plotting \( v \) against time \( t \) and fitting it with a straight line.

A demonstration example is illustrated in Fig. 1. A single spherical bubble was placed in a cylindrical domain and started to rise from rest (Fig. 1a). The time courses of the individual forces acting on the bubble are displayed in Fig. 1b. The buoyancy \( F_b \) and added mass \( F_a \) forces are given above. The drag force was evaluated as \( F_d = m_p a - F_b + F_a \), where \( m_p a \) is denoted as the inertia force \( F_i \). Fig. 1c gives details of these forces during the short initial interval of
10^{-5}\text{s}. The AM force was roughly constant over this initial interval and the drag was negligible, being less than 3\% of the AM force. The bubble speed is plotted in Fig. 1d, its slope representing the acceleration. And the acceleration perpendicular to gravity sketched in Fig. 8a was less than 6\% of the main acceleration (\(T = 5\text{s}\)). The AM force was roughly constant over this initial interval and the drag was negligible, being less than 3\% of the AM force.

Over such small time scales the bubble shape remained spherical. For instance, the period of the first mode of bubble shape oscillations was short, \(T \approx 2\text{s}\), which in our case gives \(T_2 \approx 0.008\text{s}\). Consequently, as both \(T_1\) and \(T_2\) are larger than \(10^{-5}\text{s}\), bubble deformation is insignificant.

The tensorial nature of \(C\) was not addressed in this study. Instead, we focused on the value of \(C\) along the dominant direction of motion. Nevertheless, in few cases, the values of the other components of \(C_{ij}\) were also calculated to estimate their magnitude. For example, the lateral acceleration of the bubble in Fig. 7 was found to be 2–3 orders of magnitude smaller than the main acceleration. And the acceleration perpendicular to gravity sketched in Fig. 8a was less than 6\% of the main acceleration (\(h/r = 1.1, \alpha = 45^\circ\)).

With these arrays, which consisted of multiple bubbles, all bubbles were released from rest in stagnant fluid simultaneously and the individual accelerations determined. The values of \(C\) obtained in these cases may not correspond to those actually occurring in the complex flow situations of multiphase mixtures, but are considered good enough to serve as a first estimate.

3. Numerical simulations

Unless otherwise stated, all simulations were performed using Fluent 6.1 software [Fluent 6.1 User Guide 2003]. To simulate bubble motion, we used the VOF model in which the evolution of the bubble interface is described by an advection equation. This model is designed for two or more immiscible fluids where the position of the interface between these fluids is of interest. A single set of continuity and momentum equations is shared by the fluids:

\[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.\]  \hspace{1cm} (3.1)
\[\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = - \nabla p + \nabla \cdot [\mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + \rho \mathbf{g} + \mathbf{S}.\]  \hspace{1cm} (3.2)

For each phase, volume fraction \(\epsilon_q\) of phase \(q\) is introduced and serves as a marker function in the computational cell. In each control volume, the volume fractions of all phases combined sum to unity. For example, the lateral acceleration of the bubble in Fig. 7 was found to be 2–3 orders of magnitude smaller than the main acceleration. And the acceleration perpendicular to gravity sketched in Fig. 8a was less than 6\% of the main acceleration (\(h/r = 1.1, \alpha = 45^\circ\)).

With these arrays, which consisted of multiple bubbles, all bubbles were released from rest in stagnant fluid simultaneously and the individual accelerations determined. The values of \(C\) obtained in these cases may not correspond to those actually occurring in the complex flow situations of multiphase mixtures, but are considered good enough to serve as a first estimate.

The interface motion is inferred indirectly from the motion of phases and is described by the advection equation for the volume fraction \(\epsilon_q\):

\[\frac{\partial \epsilon_q}{\partial t} + \nabla \cdot (\epsilon_q \mathbf{u}) = 0.\]  \hspace{1cm} (3.3)

Ideally, volume fraction changes discontinuously from 1 to 0 at the interface. The discretizations and approximations used in the VOF
calculations lead to interface smearing, but this can be minimized using various techniques.

Surface tension effects are modelled using the continuous surface force model (Brackbill et al., 1992), in which the surface force is represented by a smoothly varying volumetric force \( \mathbf{S} \) in Eq. (3.2) that acts on all fluid elements in the interface region, \( \mathbf{S} = 2 \sigma \mathbf{n} \mathbf{n} \cdot \nabla \mathbf{q} / (\rho q + \rho p) \). In this model, the surface curvature is \( \mathbf{n} = \mathbf{V} / \mathbf{V} \cdot \mathbf{n} \), the unit normal is \( \mathbf{n} = \mathbf{V} / \mathbf{V} \cdot \mathbf{n} \) and the normal is \( \mathbf{n} = \mathbf{V} / \mathbf{V} \cdot \mathbf{n} \).

The PRESTO! discretization scheme was used for pressure, the QUICK algorithm for momentum and the PISO algorithm for the calculation of velocities set to zero. The values of key parameters used in our simulations are shown in Table 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>CPD in bubble region (dimensionless)</th>
<th>Timestep (s)</th>
<th>( \rho_l ) (kg m(^{-3}))</th>
<th>( \rho_G ) (kg m(^{-3}))</th>
<th>( \mu_l ) (kg m s(^{-1}))</th>
<th>( \mu_G ) (kg m s(^{-1}))</th>
<th>( \sigma ) (N m(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>80</td>
<td>10(^{-6})–5 \times 10(^{-5})</td>
<td>1000</td>
<td>1000</td>
<td>0.001003</td>
<td>0.001003</td>
<td>0.073</td>
</tr>
<tr>
<td>B</td>
<td>80</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073–0.073</td>
</tr>
<tr>
<td>C</td>
<td>80</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>D</td>
<td>40</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>E</td>
<td>80</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>F</td>
<td>20–20 per 2 mm</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>G</td>
<td>40</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>H</td>
<td>20–80</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.0073</td>
</tr>
<tr>
<td>I</td>
<td>80 (across b)</td>
<td>1.0(-6)</td>
<td>998.2</td>
<td>1.225</td>
<td>0.01003</td>
<td>1.7894 \times 10(^{-5})</td>
<td>0.00073</td>
</tr>
</tbody>
</table>

Nine different cases were employed (cases from A to I). CPB: number of grid ‘cells per bubble’; numerical time step; \( \rho_l \): liquid density, \( \rho_G \): gas density, \( \mu_l \): liquid viscosity, \( \mu_G \): gas viscosity, \( \sigma \): surface tension.

In terms of the initial conditions, bubbles of the prescribed shape (2 mm spherical domain) was prescribed between the periodic boundaries. Drop of \( \Delta p \) (e.g., 1.3% for a 2 mm spherical bubble and configuration were placed into computational domains and all simulations were derived for potential flow as a classical result (e.g., Lamb, 1932).

In agreement with the literature, a value of 0.5 was analytically derived for potential flow as a classical result (e.g., Lamb, 1932; Laurien and Niemann (2004) reproduced this value by CFD simulation, calculating \( C \) from the initial acceleration.

4. Results and discussion

4.1. Single bubble in an unbounded domain

4.1.1. Spherical bubble

The AM for a single spherical bubble in an infinite domain was obtained for a sufficiently large finite domain, approaching \( C_0 = 0.5 \) as the distance grew from the walls or the other bubbles. This was subsequently used as a reference value. Thus, for example, in the case of a bubble in a spherical domain (Section 4.6.2), \( C = 0.500 \) when the domain-to-bubble diameter ratio \( D/d = 8 \). Or, for an infinite row of bubbles in a tube (Section 4.5), \( C = 0.49 \) for the largest tube diameter and bubble spacing considered.

In agreement with the literature, a value of 0.5 was analytically derived for potential flow as a classical result (e.g., Lamb, 1932; Laurien and Niemann (2004) reproduced this value by CFD simulations, calculating \( C \) from the initial acceleration.

4.1.2. Ellipsoidal bubble

The definition sketch for this arrangement is shown in Fig. 2a and the results are given in Tables 2 and 3 and Fig. 2b. The 2D axisymmetric cylindrical coordinates were used. The relevant parameters are specified under Case F in Table 1.

The domain size was 20 × 20 mm. Grid independence was tested, with both coarse (\( \delta = 0.1 \mathrm{mm} \)) and fine (0.05 mm) grids giving reasonable results (Table 2). The effect of surface tension on the fine grid was investigated, and found to be negligible because the deformation time scale was longer than the CFD run (\( \sim 10^{-5} \) s) (see Table 3).

The bubble aspect ratio, \( E = a/b \), ranged from 1:4 (oblate) to 4:1 (prolate). The main result was that \( E \) increases as the bubble becomes more flat (Fig. 2b—marks), recovering \( C_0 \) at spherical shape \( E = 1 \).

The following analytical solution exists for an ellipsoid in infinite fluid moving along the direction of one of its axes (Lamb, 1932; Milne-Thomson, 1968, see also Kendouss, 2007):

\[
C_e = Z/(2 - Z), \quad Z = abc \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}, \quad k^2 = \left( a^2 + \lambda \right) \left( b^2 + \lambda \right) \left( c^2 + \lambda \right).
\]

(4.1.1)
Fig. 2. Ellipsoidal bubble in unbounded domain (Section 4.1.2). (a) Definition sketch; horizontal semi-axes coincide \((b = c)\), aspect ratio is \(E = a/b\). (b) Result: added mass \(C\) decreases with bubble aspect ratio \(E\); CFD simulation (marks), theoretical solution by Eq. (4.1.1) (line).

Table 2
Ellipsoidal bubble in unbounded domain (Section 4.1.2)

<table>
<thead>
<tr>
<th>(a) (mm)</th>
<th>(b) (mm)</th>
<th>(C) Coarse grid</th>
<th>(C) Fine grid</th>
<th>(C) Analyt.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2</td>
<td>2.341</td>
<td>2.374</td>
<td>2.374</td>
</tr>
<tr>
<td>0.7</td>
<td>1.5</td>
<td>1.193</td>
<td>1.2</td>
<td>1.204</td>
</tr>
<tr>
<td>0.9</td>
<td>1.1</td>
<td>0.636</td>
<td>0.635</td>
<td>0.634</td>
</tr>
<tr>
<td>1.1</td>
<td>0.9</td>
<td>0.397</td>
<td>0.393</td>
<td>0.392</td>
</tr>
<tr>
<td>1.5</td>
<td>0.7</td>
<td>0.196</td>
<td>0.193</td>
<td>0.192</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.085</td>
<td>0.083</td>
<td>0.082</td>
</tr>
</tbody>
</table>

Effect of computational grid on added mass \(C\), simulations vs. analytical solution by Eq. (4.1.1) coarse grid \(\delta = 0.1\) mm, fine grid \(\delta = 0.05\) mm (see Fig. 2a).

Lamb (1932) gives the closed formulas:

\[
C = \frac{E \cos^{-1} E - \sqrt{1 - E^2}}{E^2 \sqrt{1 - E^2} - E \cos^{-1} E} \quad (E < 1) \tag{4.1.2}
\]

and

\[
C = \frac{E \ln(E + \sqrt{E^2 - 1}) - \sqrt{E^2 - 1}}{E^2 \sqrt{E^2 - 1} - E \ln(E + \sqrt{E^2 - 1})} \quad (E > 1) \tag{4.1.3}
\]

Our calculations agreed with this analytical solution (Fig. 2b—marks vs. line).

4.1.3. Spherical cap bubble

The definition sketch is in Fig. 3 and the results are in Tables 4 and 5.

Three simulations were performed for a semi-spherical bubble with a radius \(r\) of 1 mm. CFD1 involved using VOF to accelerate a fluid particle (domain size \(8 \times 8\) mm; parameters corresponding to case \(H\) in Table 1) and gave \(C = 0.852\) (Table 4). The results were relatively grid independent (Table 5). Because a large surface force \(S\) (last term in Eq. (3.2)) was generated by the curvature at the bubble...
edges, a very low surface tension had to be used ($\sigma = 0$) to secure grid independence.

CFD2 was a stationary rigid semi-spherical bubble measured in an accelerating flow field with a domain size of $16 \times 32$ mm. The AM force was calculated, yielding $C = 1.24$.

CFD3 was a rigid body accelerated from rest in stagnant fluid (domain size $16 \times 32$ mm). The AM force was then calculated, yielding $C = 1.22$.

Although we are not currently able to explain the variance between these three results, the fact that a difference exists simultaneously serves as both a warning and a challenge for further studies. All three methods yielded the same result, $C = 0.5$, for a fully spherical particle.

For an approximate inviscid solution (potential flow ahead of the bubble, Hill’s vortex behind it), Kendoush (2003) reported that AM increases rapidly with bubble flatness: $C = \infty$ for a flat disk; $C = 2.04$ for a half-sphere.

### 4.1.4. 2D bubble

Simulations for a 2D circular bubble (infinite cylinder) in unbounded fluid were performed using 2D Cartesian coordinates. A single bubble ($d = 2$ mm) was placed in the middle of a $20 \times 20$ mm square domain (effectively ‘unbounded’), the parameters corresponding to case E in Table 1. Two grids were tested and $C = 1$ was obtained for each. These results agree with various analytical solutions (e.g. Keulegan and Carpenter, 1958; Newman, 1977) (see Table 6).

### 4.2. Single bubble and wall

#### 4.2.1. Perpendicular motion

**Spherical bubble.** The definition sketch is in Fig. 4a, and the results are in Table 7 and Fig. 4b. The 2D axisymmetric cylindrical coordinates were used. The relevant parameters correspond to case E in Table 1.

The domain size was $20 \times 20$ mm, the bubble radius $r = 1$ mm and the wall distance $h/r$ ranged from 1.1 to 5.0. A relatively fine grid was used (Table 7). The surface tension was maintained at the low level of 0.00073 N/m. The effect of the domain size on $C$ was small (errors less than 1%). The main result observed was that the value of $C$ fell quickly as the wall distance grew (Fig. 4b—marks), the influence of the wall only being significant at a very short distance (few bubble radii), which is typical for the inertial force. At $h/r = 3$, $C$ differs from $C_0$ by only 1.5%.

Milne-Thompson’s (1968) approximate analytical solution (first order method of images) states that

$$C_{pe} = C_0(1 + (3/8)r^3/h^3).$$ (4.2.1)

Our calculations agreed with this leading order solution (Fig. 4b—marks vs. line).

This case is equivalent to two bubbles moving along a line of centres due to the plane of symmetry. Further terms in the two-bubble expansion have been published by Kok (1993), see also Legendre et al. (2005): $1 + (3/8)\frac{r^3}{h^3} + (3/64)\frac{r^6}{h^6} + (9/256)\frac{r^9}{h^9} + (3/512)\frac{r^{12}}{h^{12}} + \ldots$, which applies also to $C_{pe}$.
4.2.1.2. Ellipsoidal bubble. Definition sketch is in Fig. 5 and the results are given in Fig. 6. In this case Fluent ver. 6.3 was used. 2D axisymmetric cylindrical coordinates were used, the parameters corresponding to case I in Table 1.

The domain size was 16 × 16 mm. One semi-axis was fixed (b = 1 mm), while the other semi-axis (a) varied in length. The bubble distance h/a ranged from 1.1 to 10 at fixed aspect ratio E = a/b = 1/2. E ranged from 1/5 to 2/3 at a fixed distance h/a = 2. The effect of the domain size on C was small. The main result being that the value of C fell with both h and E (Fig. 6—marks).

As we were unable to find an analytical solution for this case in the literature, we suggest the following empirical formula, which preserves the structure of Eq. (4.2.1):

\[ C = C_0(1 + (3/8)(x^2/h^3)) \]

In this formula, the spherical coefficient C0 is replaced by the ellipsoidal one C given by Eq. (4.1.1) in which r was replaced by another length scale defined as \( r = bh/(h - a) \). This resulted in a good data fit with Eq. (4.2.2). Note that \( x \) approaches \( a \) on contact (h = a) and recovers Eq. (4.2.1) at \( a = b = r \) for spherical shapes. For a flat bubble \( E = 1/5, h/a = 2 \) and close spacing \( h/a = 1.1, E = 1/2 \), Eq. (4.2.2) fits the data with an error of about 10%, which rapidly fell as both h and E increased.

4.2.2. Parallel motion of spherical bubble

Definition sketch is in Fig. 7. The 3D Cartesian simulations were performed, the parameters corresponding to case G in Table 1.

The wall distance was fixed at \( h/r = 1.1 \). The bubble radius was \( r = 1 \) mm. The resulting value of C was 0.558, which is in good agreement with Milne-Thompson’s (1968) analytical solution, where

\[ C_{pa} = C_0(1 + (3/16)(r/h)^3) \]

yields \( C = 0.570 \).

This case is equivalent to two bubbles moving perpendicular to a line of centres, due to symmetry.

4.2.3. Oblique motion of spherical bubble

Definition sketch is in Fig. 8a and the results are given in Fig. 8b. The 3D Cartesian simulations were performed, the parameters corresponding to case G in Table 1.

A cubed domain was used (16 × 16 × 16 mm). The bubble, with a fixed radius \( r = 1 \) mm, was positioned near one of the cube walls. The wall distance was fixed at \( h/r = 1.1 \) (close spacing). The angle of approach \( x \) ranged from 0° (perpendicular) to 90° (parallel) by changing the direction of gravity. The computational grid consisted of several zones, the finest grid being near the bubble and the coarser grids further away (Table 8). The effect of the domain size on C was small. The main result observed was the dependence of C on \( x \) (Fig. 8b).

As we could not find an analytical solution for this case in the literature, we propose the following empirical formula:

\[ C = A\cos(2x) + B, \quad A = (C_{pe} - C_{pa})/2, \quad B = (C_{pe} + C_{pa})/2. \]

where \( C_{pe} \) and \( C_{pa} \) are the limits of perpendicular motion \( (x = 0°) \) and parallel motion \( (x = 90°) \) given by Eqs. (4.2.1) and (4.2.3), respectively. At these limits, the numerical results agree with the analytical solution within 3%. This case is equivalent to two bubbles mirrored by the plane of symmetry.

4.3. Two spherical bubbles in unbounded domain

Definition sketch is in Fig. 9, results in Figs. 10–12. The 3D Cartesian simulations were performed, the parameters corresponding to case G in Table 1.

Two spherical equal-sized bubbles of fixed diameter \( (d = 2 \) mm) were positioned inside a cubed domain \( (27.2 \times 27.2 \times 27.2 \) mm\(^3\)). The bubble distance \( D/d \) ranged from 1.1 to 5.0 and the angle of alignment \( x \) ranged from 0° (vertical) to 90° (horizontal). To save
processing time and power, the simulations were initially performed for one bubble only in order to determine a suitable grid and software configuration. A fine grid was used up to 1.5 bubble radii from the bubble surface. Beyond this, coarser grids were used (see Table 8). Forty CPB ensured acceptable grid independence. The effect of the domain size on $C$ was small. Due to the fore–aft symmetry of potential flow, the same value of $C$ was obtained for both bubbles.

### 4.3.1. Vertical alignment
The AM of two bubbles in-line ($\alpha = 0^\circ$) decreases because this doublet is more streamlined than a single sphere (Fig. 10a). However, due to the short-range character of inertia forces, this decrease is only evident with close spacing (a few radii). The following empirical formula was developed to fit our CFD data:

$$C_v = 0.35 + 0.15(D/d - 1)/(D/d - 0.546), \quad (4.3.1)$$

giving 0.35 for touching bubbles and 0.5 where there was no contact. Our results are in relatively good agreement with those of Helfinstine (1974), who used the method of images (Fig. 10a). Note that this arrangement is different from a bubble perpendicular to the wall in Section 4.2.1 (cf. Figs. 4b and 10a).

### 4.3.2. Horizontal alignment
The AM of two bubbles side by side ($\alpha = 90^\circ$) is greater (Fig. 10b) because this doublet is less streamlined than a single sphere. Again, this effect only applies at short range. At a $D/d$ of about 2–3, AM approaches a constant value, but slightly underpredicts the reference value $C_0$ expected for wide spacing. The following empirical formula was developed to fit our CFD data:

$$C_h = 0.61 - 0.11(D/d - 1)/(D/d - 0.862), \quad (4.3.2)$$

giving 0.61 for touching bubbles and 0.5 where there was no contact. Again, our results compare relatively well with the approximate solution of Helfinstine (1974), who used the method of images (Fig. 10b).

Note that this arrangement is equivalent to a bubble rising along a wall (Section 4.2.2). Consequently, the case where $h/r = 1.1$ and $C = 0.558$ in Section 4.2.2 corresponds to $D/d = 1.1$ and $C = 0.558$ in Fig. 10b (black marks). Ideally, $C_{pa} = C_h$ and therefore, the Eqs. (4.2.3) and (4.3.2) are equivalent. In our simulations, we encountered some discrepancies, the magnitude of which can be used to assess possible errors connected with these types of calculation.

### 4.3.3. General position
The effect of the inclination angle $\alpha$ is demonstrated in Fig. 11 (marks) for fixed bubble spacing $D/d = 1.5$. The AM gradually increases from the limit of the vertical pair ($C \approx 0.44$) to the limit of the horizontal pair ($C \approx 0.52$), passing through the reference value $C_{pa}$.

---

![Fig. 8](image-url) Spherical bubble at inclined wall (Section 4.2.3). (a) Definition sketch; bubble size fixed to $r = 1$ mm, wall distance fixed to $h/r = 1.1$ (close to wall), approach angle $\alpha$ varies. (b) Result: added mass $C$ decreases with $\alpha$; CFD simulation (black marks), empirical formula by Eq. (4.2.4) (line), two limits $\alpha = 0^\circ$ and $90^\circ$ correspond to $C_{pe}$ and $C_{pa}$ by Eqs. 4.2.1 and (4.2.3) (white marks).

---

![Fig. 9](image-url) Two bubbles in general position (Section 4.3). Definition sketch; bubble size fixed to $d = 2$ mm; bubble distance $D$ and alignment angle $\alpha$ vary.
\[
C_0 = 0.5 \text{ at about } \alpha \approx 60^\circ. \text{ The following empirical formula was developed to fit our CFD data:}
\]
\[
C = A \sin(2(\alpha - 45^\circ)) + B, \quad A = \frac{1}{2}(C_V - C_H),
\]
\[
B = \frac{1}{2}(C_V + C_H),
\]
(4.3.3)

where \(C_V\) and \(C_H\), the aforementioned vertical and horizontal coefficients, are dependent on distance \(D/d\) (Fig. 11—dashed line). Our results are in relatively good agreement with the approximate solution published by Kok (1993), who considered the first four terms in the method of images (Fig. 11—full line). The differences between our results and Kok’s are more pronounced at low \(\alpha\) and \(D/d\). For completeness, Fig. 12 shows the influence of both \(D/d\) and \(\alpha\) on \(C\). Their combined effect on AM can be clearly seen, causing both a decrease and an increase in the value of \(C\). Similar behaviour has been reported for drag and lift (Legendre et al., 2003).

### 4.4. Three spherical bubbles

Definition sketch is shown in Fig. 13 and the results in Fig. 14. The 3D Cartesian simulations were performed with parameter values corresponding to case G in Table 1.

The computational domain was a 27.2 mm cube. Three bubbles (\(d = 2\) mm) forming an equilateral triangle were placed in the centre of the domain in one of two different configurations, horizontal (Fig. 13a) or vertical (Fig. 13b). The horizontal bubbles behaved identically due to symmetry (finite domain shape was assumed to have no effect), whereas the vertical cluster consisted of one leading bubble (1) and two identical trailing bubbles (2, 3). The bubble distance \(D/d\) ranged from 1.1 to 2.0. The computational grid was finer closer to the bubbles and coarser further away (see Table 8). In both cases, the effect of the domain size on \(C\) was small.

In the horizontal triangle, each bubble had the same value of \(C\), which was larger than both \(C_0\) and \(C_H\) (Fig. 14a). The flat triangle was much less streamlined. This result clearly demonstrates that the presence of other bubbles can strongly affect inertial coupling based only on pairwise interactions. As such multiple interactions are difficult to obtain both analytically and experimentally, and the CFD approach can be of great help.

The results obtained for the vertical triangle can be summarized as \(C_1 < C_2 < C_3 < C_0\) (Fig. 14b—marks). The value of \(C_1\) is close to that of \(C_0\) obtained for the vertical doublet. The \(C_{2,3}\) values fall between the values of \(C_2\) and \(C_3\) obtained for the vertical and horizontal doublets, and were close to a doublet inclined at \(\alpha \approx 50^\circ\). Part of the vertical triangle (bubbles 1 and 2) can be considered as a doublet inclined at \(\alpha \approx 30^\circ\). For comparison purposes, Fig. 14b also plots the corresponding line obtained using Eq. (4.3.3), this line passing between the lines for \(C_1\) and \(C_2\).

We believe that horizontal and vertical limits on bubbles in triangular configurations set bounds on the value of \(C\) between these two limits. Our results for three bubbles also indicate the relevance of multiple inertial coupling in bubbly mixtures.

### 4.5. The 1D array of spherical bubbles in a tube

Definition sketch is shown in Fig. 15 and the results are given in Tables 9–12 and Fig. 16. The 2D axisymmetric cylindrical coordinates were used with the parameters corresponding to case C in Table 1 (main results).
The free-slip boundary condition was used at the tube wall; the periodic boundary condition along the tube axis. The bubble radius was fixed at 1 mm. The tube radius ranged from 1.1 to 20. By altering the value of $H$, the bubble spacing $H/r = 2H/d$ ranged from 1.1 to 20; $H$ being correspondingly enlarged as the number of bubbles in the domain increased ($N = 1, 2$, etc.). Four different grids were used within a range of 20–160 CPB. The grid test showed that 80 CPB was acceptable (Table 9). Changing the viscosity (by four orders)
and surface tension (by three orders) had virtually no effect on the computation results (Tables 10 and 11). As the number of bubbles in the domain did not affect the results (Table 12), \( N = 1 \) was used in all calculations.

At fixed tube size \( R \), \( C \) increases with bubble spacing \( H \) (Fig. 16a—black marks). As a dense array becomes more streamlined, \( C \) decreases. To provide a comparison, the data for the unbounded vertical doublet \( C_V \) (from the simulation depicted in Fig. 10a) are also shown (white marks). At wider spacing \( H \), the rise in the value of \( C \) above 0.5 is due to the confining tube wall. At fixed bubble spacing \( H \), \( C \) decreases with tube size \( R \), which is similar to a bubble moving parallel to the wall (Fig. 16b—black marks). In the tube, the value of \( C \) is higher due to the presence of the wall. At high values of both \( H \) and \( R \), the reference value \( C_0 \) is recovered.

The important consequence of these results is that \( C \) is dependent on the aspect ratio of the domain. For narrow and tall domains, the wall effect prevails and AM increases \((C > 0.5)\), whereas for wide and short domains, the array effect prevails and AM decreases \((C < 0.5)\). Obviously, these effects mutually cancel themselves out at a certain domain proportion \((C = 0.5)\).

Our simulation results were compared with Cai and Wallis (1992) approximate solution (Fig. 16—black lines). For potential flow past an infinite 1D array of equidistant spheres in a circular open-ended tube, they obtained

\[
C = (\beta - 1)HR^2/(2(3)/\beta) - 1. \tag{4.5.1}
\]

For \( 1 < H/r < 6 \) and \( 1 < R/r < 2.5 \), the parameter \( \beta \) was correlated as

\[
\beta = 1 + a(H/r)^b.
\]

\[
a(R/r) = \exp(1.3916 - 7.2445\ln(R/r) + 7.049(\ln(R/r))^2
- 3.3251(\ln(R/r))^3), \tag{4.5.2}
\]

\[
b(R/r) = 0.99832(R/r)^{-0.084924}.
\]

Good agreement was found between their results and ours. An array in one tube can be used as an ‘elementary cell’ in dispersed flow, possibly approximating to an ensemble of ‘bubble chains’ with a spacing of 2H in the vertical direction and of 2R in the horizontal direction. Qualitatively similar trends also apply to arrays in channels with non-circular cross sections.

### 4.6. The 3D bubble arrays

If real dispersions are to be correctly modelled in general and interphase inertial coupling to be determined in particular, the effects of AM must be clearly understood. An important part of this involves understanding how \( C \) is dependent on bubble volume fraction and configuration. Certain important details of bubble arrangements are not always resolved, often being considered ‘random’ or ‘regular’.

Typically AM increases with bubble concentration \( c \). Some results have been obtained using simple geometric concepts. For example, Zuber (1964) applied the classical formula for a spherical body in a spherical domain (e.g., Lamb, 1932),

\[
C = C_0(1 + 2(d/D)^3)/(1 - (d/D)^3), \tag{4.6.1}
\]

and considering the domain as an elementary cell in the dispersion of concentration \( c = (d/D)^3 \), obtained

\[
C = C_0(1 + 2c)/(1 - c). \tag{4.6.2}
\]
This cell model is relatively robust, works for all values of $c$ and gives reasonable results. At the dilute limit, the expansion of Eq. (4.6.1) yields

$$C = C_0 (1 + 3c) + O(c^2).$$

(4.6.3)

Other results have been obtained on theoretical grounds using demanding analytical efforts based on averaging techniques or energy concepts. These formulas usually take the form

$$C = C_0 (1 + kc) + O(\text{higher terms}),$$

(4.6.4)

where the value of $k$ is typically close to 3, but varies according to the particular approach used, the specific assumptions made and the bubble configurations, velocity distributions, etc. employed. For instance, Wijngaarden (1976) obtained $k=2.78$ while both Kok (1988) and Biesheuvel and Spoelstra (1989) reported $k = 3.32$. Spelt and Sangani (1998) used the averaged equations of motion of bubbly mixtures at large $Re$ and small $We$, solved them numerically, and reported $k = 3$. Sankaranarayanan et al. (2002) used a two-continua model within the Lattice Boltzmann approach to simulate bubble arrays in a periodic box. For spherical and distorted bubbles at finite $Re$, they evaluated $C$ from the increase in bubble speed due to increased gravity, and obtained $k=4$ as a compromise for most of their data. Laurien and Niemann (2004) and Niemann and Laurien (2001) employed the DNS approach and calculated $C$ from initial bubble acceleration. For a single bubble, a value of 0.5 was correctly obtained for $C_0$. Using a periodic box, for a regular array they found that

$$C = C_0 (1 + 3.26c + 7.70c^2),$$

(4.6.5)

where $k=3.26$, which is in agreement with the analytical results.

Alternatively, the generalized cell model used by Cai and Wallis (1994),

$$C = [(1 - c) + k(0.5 + c)]/[(2 + c) + k(1 - c)],$$

(4.6.6)

predicts both an increase and decrease in AM with bubble concentration $c$ depending on the value of parameter $k$. For $k=\infty$, their model recovers the increasing Zuber result (Eq. (4.6.2)). For $k=0$, it gives a decreasing result:

$$C = (1 - c)/(2 + c),$$

(4.6.7)
with the dilute limit
\[ C = C_0(1 - 1.5c) + O(c^2). \]  
\text{(4.6.8)}

The increase in AM at large \( k \) corresponds to a fixed outer shell (stationary rigid wall, Eq. (4.6.2)). The decrease in AM at low \( k \) corresponds to a movable or penetrable outer shell (ideal pressure release surface, Eq. (4.6.7)). Parameter \( k \) can be seen as the ratio of (outer)/(inner) fluid density.

### 4.6.1. Spherical bubble in a periodic box

**Definition sketch** is shown in Fig. 17a and the results in Fig. 17b. The 3D Cartesian simulations were performed and the parameters corresponding to case G is given in Table 1.

The bubble size was fixed at \( d = 2 \text{ mm} \). The periodic box size \( D/d \) ranged from 1.1 to 6 by altering the value of \( D \). The finer grid (CPB = 40) was closer to the bubble, the coarser grid further away (Table 8). Three directions of bubble motion (gravity) were tested: box face (\( g_1 \)), edge (\( g_2 \)) and corner (\( g_3 \)). The direction of gravity had no effect on the CFD results (Fig. 17b). This situation corresponds to

![Spherical bubble in spherical domain (Section 4.6.2)](image)

**Fig. 18.** Spherical bubble in spherical domain (Section 4.6.2). Definition sketch; bubble size fixed to \( d = 2 \text{ mm} \), domain size \( D \) varies. Situation corresponds to 3D random array with bubble concentration \( c = (d/D)^3 \).

#### Table 13

<table>
<thead>
<tr>
<th>CPD</th>
<th>( D ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2</td>
<td>2.04</td>
</tr>
<tr>
<td>2.5</td>
<td>1.061</td>
</tr>
<tr>
<td>3</td>
<td>0.667</td>
</tr>
<tr>
<td>4</td>
<td>0.595</td>
</tr>
<tr>
<td>5</td>
<td>0.567</td>
</tr>
<tr>
<td>6</td>
<td>0.504</td>
</tr>
<tr>
<td>8</td>
<td>0.491</td>
</tr>
<tr>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>

Grid refinement study: effect of CPB and domain size \( D \) on resulting value of \( C \), analytical solution by Eq. (4.6.1) (see Fig. 18).

#### Table 14

<table>
<thead>
<tr>
<th>( \mu_L ) (kg/ms)</th>
<th>( C ) (dimensionless)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01003</td>
<td>0.702</td>
</tr>
<tr>
<td>0.001</td>
<td>0.7</td>
</tr>
<tr>
<td>0.00001</td>
<td>0.7</td>
</tr>
<tr>
<td>Inviscid</td>
<td>0.698</td>
</tr>
<tr>
<td>Analytical solution</td>
<td>0.714</td>
</tr>
</tbody>
</table>

Effect of liquid viscosity \( \mu_L \) on resulting value of \( C \) (\( D = 4 \text{ mm} \), \( d = 2 \text{ mm} \), CPB = 80), inviscid solution for \( \mu_L = 0 \), analytical solution by Eq. (4.6.1). (see Fig. 18).

![Spherical bubble in spherical domain (Section 4.6.2). Result: added mass \( C \) increases with bubble concentration \( c \) (marks) (cf. Figs. 20 and 17). For comparison: Zuber cell model by Eq. (4.6.2) (bold line), its linearization by Eq. (4.6.3) (thin line), Laurien and Niemann result for periodic box by Eq. (4.6.5) (dashed line).](image)

**Fig. 20.** Spherical bubble in spherical domain (Section 4.6.2). Result: added mass \( C \) increases with bubble concentration \( c \) (marks) (cf. Figs. 20 and 17). For comparison: Zuber cell model by Eq. (4.6.2) (bold line), its linearization by Eq. (4.6.3) (thin line), Laurien and Niemann result for periodic box by Eq. (4.6.5) (dashed line).
an infinite 3D regular bubble array where \( c=(\pi/6)D^3 \) and ranged from 0.002 to 0.393. The periodic boundary condition mimicked the pressure release surface, rather than the fixed wall.

Contrary to general expectation, we found that \( C \) decreases with bubble concentration (Fig. 17b). The reason for this is as follows. Firstly, the periodic box is close to pressure release case, so that, in principle, AM can decrease (see Eq. (4.6.7)). Secondly, the periodic box \( H \times D \times D \) is equivalent to a 1D bubble array of spacing \( H \) in a rectangular channel of cross-section \( D \times D \). We know that \( C \) increases with \( H \) (Fig. 16a) and decreases with \( D \) (Fig. 16b). Thus for a tall domain \( C \) will increase, while for a wide domain it will decrease. With our isometric domain \( H=D \), the decreasing tendency prevails. Fig. 17b also plots Cai and Wallis’s (1994) results (Eqs. (4.6.7) and (4.6.8)) and shows that their data agrees with ours in this case.

The following empirical formula was developed to fit our CFD data (see Fig. 17b):

\[
C = C_0(1 - 1.048c - 0.588c^2).
\]

Laurien and Niemann (2004) performed the same calculations, but obtained an increase (Eq. (4.6.5)). The reason for this discrepancy may be their use of a tall anisometric domain where \( H = 5D \) (see their Figs. 8 and 9).

4.6.2. Bubble in a spherical domain

4.6.2.1. Spherical bubble. Definition sketch is shown in Fig. 18 and the results are given in Tables 13 and 14 and Figs. 19 and 20. 2D axisymmetric coordinates were used, with the parameters corresponding to case B in Table 1 (80 CPB are used for the main results; CPB 20–160 are used for the grid independence study).

The free-slip boundary condition was used at the wall. The bubble size \( d \) was fixed at 2 mm. By modifying \( D \) the shell size \( D/d \) ranged from 1.1 to 8. The grid independence study showed that where CPB = 80 the results can be considered grid independent; the difference between 80 and 160 CPB being less than 1% except for \( D/d=2.5 \) (about 5%) (Table 13). The influence of liquid viscosity on the simulations was negligible (Table 14). Due to the formulation of the volumetric surface force in VOF (Fig. 19), a realistic surface tension of 0.073 N/m caused problems for the small domains (\( D<5 \) mm). It was, therefore, reduced by a factor of 10–100 to secure grid independence without affecting the output. This situation corresponds to that of an infinite 3D random bubble array where \( c=(d/D)^3 \) and ranged from 0.002 to 0.75. The main result was that \( C \) increases with bubble concentration (Fig. 20), thereby, as expected, conforming to the trend to increase. Agreement with the Zuber cell model was good.

4.6.2.2. Ellipsoidal bubble. Definition sketch and results are presented in Fig. 21. The 2D axisymmetric coordinates were used with the parameters corresponding to case I in Table 1.

The bubble size was fixed: \( a=0.5 \) mm; \( b=1 \) mm; aspect \( E = a/b = 0.5 \). By altering \( R \) the shell size \( R/a \) ranged from 1.5 to 8. The numerical parameters were the same as in the case of the spherical bubble described above. These simulations were performed using Fluent ver. 6.3. This situation corresponds to that of an infinite 3D random bubble array where \( c=ab^2/R^3 \) and ranged from \( \approx 0.002 \) to \( \approx 0.3 \).

The main result was that the value of \( C \) was higher than for the spherical bubble, increasing more quickly with bubble concentration \( c \) (Fig. 21b). We were unable to find an analytical solution for this case. The following empirical formula based on the cell model (Eq. (4.6.2)) was developed to fit our CFD data:

\[
C = C_0(1 + 2c)/(1 - c)
\]

which can be used for moderate values of \( c \). When \( E=0.5 \) and \( c \leq 0.2 \), errors of approximately only \( \approx 1\% \) are obtained. For flatter bubbles, the error is larger.

5. Conclusions

The added mass coefficient \( C \) of dispersed particles was calculated using computational fluid dynamics (CFD). As our interest is primarily in gas–liquid systems, the calculations were performed for various configurations of gas bubbles in liquids. The value of \( C \) was obtained from initial bubble acceleration for very short time intervals of \( \sim 10^{-5} \) s, where motion is controlled by buoyancy and inertia. In such cases, the effects of other phenomena are negligible (e.g., velocity dependent forces, bubble deformation, boundary layer development, vorticity production, etc.). Under these conditions, the difference between slip and no-slip boundary conditions is small, and the flow is effectively inviscid and potential. Consequently, our results obtained for gas bubbles in liquids also apply to drops and solids in fluids.

Several flow situations were considered with one or more bubbles in 1D, 2D and 3D configurations. In each case, the effects of both numerical and physico-chemical parameters were tested. Although
the value of $C$ should not be dependent on the material properties of the phases (e.g., viscosity and surface tension), these quantities can impact the numerical procedures included in the volume of fluid (VOF) model. For example, in some cases it is necessary to reduce the surface tension value in order to obtain grid independence and reasonable output. Also, in certain situations, our CFD results were at variance with solutions obtained by other authors. We intend to investigate the reasons for this in our future work.

Our results show that CFD is a useful tool for providing information about added mass in dispersions. With relative ease, it enables one to compute $C$ for situations inaccessible to analytical methods and experimental measurements. Where we were unable to find results for certain cases in the literature, we developed simple empirical formulas and fitted them to our CFD data. These simple formulas are suitable for further use. Particularly interesting were the results that showed that the presence of a third bubble can change the pattern of pair-wise bubble interactions that form the starting point for averaging efforts aimed at acquiring mean-field macroscopic equations for multiphase flows.

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